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A note on Bohr's phenomenon for power series

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A R T I C L E   I N F O

A B S T R A C T

Bohr's phenomenon, first introduced by Harald Bohr in 1914, deals with the largest radius r, $0 < r < 1$, such that the inequality $\sum_{k=0}^{\infty} |a_k| r^k \leq 1$ holds whenever the inequality $|\sum_{k=0}^{\infty} a_k z^k| \leq 1$ holds for all $|z| < 1$. The exact value of this largest radius known as *Bohr's radius*, which is $r_b = 1/3$, was discovered long ago. In this paper, we first discuss Bohr's phenomenon for the classes of even and odd analytic functions and for alternating series. Then we discuss Bohr's phenomenon for the class of analytic functions from the unit disk into the wedge domain $W_\alpha = \{w : |\arg w| < \pi\alpha/2\}$, $1 \leq \alpha \leq 2$. In particular, we find Bohr's radius for this class.

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1. Introduction

Given the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

its *majorant series* is defined by

$$M_f(z) = \sum_{k=0}^{\infty} |a_k| r^k, \tag{1.2}$$

where and in the sequel, $r = |z|$. By basic complex analysis, the series (1.1) and (1.2) converge or diverge on open disks simultaneously, that is, for any given R, $0 < R \leq \infty$,

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$$\left| \sum_{k=0}^{\infty} a_k z^k \right| < \infty \quad \text{for all } |z| < R$$

if and only if

$$\sum_{k=0}^{\infty} |a_k| r^k < \infty \quad \text{for all } r < R.$$

Of course, the values of functions $f(z)$ and $M_f(z)$ as well as values of certain norms of these functions may be very different. Our primary interest is the comparison of sup norms of $f(z)$ and $M_f(z)$ over the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and disks with smaller radii: $\mathbb{D}_\rho = \{z : |z| < \rho\}$, $0 < \rho < 1$. This study was initiated by Harald Bohr in 1914 [11], who proved a weaker version of the following theorem.

Theorem 1.1. *If $\left| \sum_{k=0}^{\infty} a_k z^k \right| \leq 1$ in the unit disk $\mathbb{D}$, then $\sum_{k=0}^{\infty} |a_k||z|^k \leq 1$ in the disk $\mathbb{D}_{1/3}$. The radius $\rho_b = 1/3$ is the best possible.*

Bohr proved this theorem for $|z| \leq 1/6$. The sharp version with radius $\rho_b = 1/3$, which is nowadays commonly known as the *Bohr's radius*, was proved independently by Wiener, Riesz, and Schur. For relevant references and interesting recent developments on the theme of Bohr's phenomenon we refer to papers [12, 15,9,7].

The majorant series (1.2) belongs to a very important class of series – series with non-negative terms. Yet, there is another class of series which is also very popular – the class of alternating series. Thus, for series (1.1) we define its associated *alternating series* as

$$A_f(z) = \sum_{k=0}^{\infty} (-1)^k |a_k||z|^k. \tag{1.3}$$

For alternating series we have the following counterpart of Theorem 1.1.

Theorem 1.2. *If $\left| \sum_{k=0}^{\infty} a_k z^k \right| \leq 1$ in the unit disk $\mathbb{D}$, then*

$$\left| \sum_{k=0}^{\infty} (-1)^k |a_k||z|^k \right| \leq 1 \tag{1.4}$$

in the disk $\mathbb{D}_{1/\sqrt{3}}$. The radius $r = 1/\sqrt{3}$ is the best possible.

The notion of Bohr's radius, initially defined for mappings from the unit disk $\mathbb{D}$ to itself, was generalized by some authors to classes of mappings from $\mathbb{D}$ to some other domains $G \subset \mathbb{C}$ (see [7,1,3,5]). One way for generalization is to rewrite Bohr's inequality in the equivalent form as $\sum_{k=1}^{\infty} |a_k||z|^k \leq 1 - |a_0|$. Then, the right-hand side $1 - |a_0|$ can be interpreted as the distance $dist(f(0), \partial \mathbb{D})$ from the point $f(0)$ to the boundary $\partial \mathbb{D}$ of a given domain which is here the unit disk $\mathbb{D}$. In this form, the notion of Bohr's radius can be generalized to the class of functions $f(z)$ analytic in $\mathbb{D}$, which take values in a given domain G as follows:

For a given domain $G \subset \mathbb{C}$, find the largest radius $r_G > 0$ such that

$$dist(M_f(z), |f(0)|) = \sum_{k=1}^{\infty} |a_k||z|^k \leq dist(f(0), \partial G) \tag{1.5}$$

for all $|z| \leq r_G$ and all functions $f(z)$ analytic in $\mathbb{D}$ and such that $f(\mathbb{D}) \subset G$.

Interestingly enough, it was shown in [7] that if G is convex then inequality (1.5) holds for all $|z| \leq 1/3$ and this radius is the best possible. Thus, if G is convex then r_G coincides with Bohr's radius $r = 1/3$ and does not depend on G. When G is any proper simply connected domain and $f(z)$ is analytic in $\mathbb{D}$ such that $f(\mathbb{D}) \subset G$, Abu-Muhanna [1] showed that (1.5) holds for all $|z| \leq 3 - 2\sqrt{2}$ and this radius is sharp for the Koebe function $k(z) = \frac{z}{(1-z)^2}$. This implies that $r_G \geq 3 - 2\sqrt{2} = 0.1715\ldots$ for any simply connected domain G.

As for non-convex domains, the authors of a recent paper [4] initiated the study of Bohr's problem for mappings from $\mathbb{D}$ into the wedge-domain $W_\alpha = \{w : |\arg w| < \frac{\pi\alpha}{2}\}$, $1 \leq \alpha \leq 2$, which is not convex except the case $\alpha = 1$ when W_α is the right half-plane. Our main result for mappings into the wedge-domain is the following theorem generalizing Theorem 2.3 in [4].

Theorem 1.3. *Let* $1 \leq \alpha \leq 2$ *and suppose* $f(z) = a_0 + \sum_{k=1}^{\infty} a_k z^k$ *maps* $\mathbb{D}$ *into* W_α. *Then*

$$dist\left(\sum_{k=0}^{\infty} |a_k||z|^k, |a_0|\right) = \sum_{k=1}^{\infty} |a_k||z|^k \leq dist(a_0, \partial W_\alpha) \tag{1.6}$$

for $|z| \leq r_\alpha$, *where* $r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$. *The radius* r_α *is the best possible.*

In the special case when $a_0 = f(0)$ is real and positive, this theorem was proved in [4].

The proof of Theorem 1.2 as well as some generalizations of this theorem are discussed in Section 2. In particular, we find Bohr's radius for the class of even functions and give lower and upper bounds for Bohr's radius for the class of odd functions. Certain portions of the proofs presented in this section took insights from [15]. We want to mention here that a paper [10] by R. P. Boas could be a good source of additional information on this topic. At the end of Section 2, we introduce *argument symmetric series*, which include majorant series and alternating series as special cases, and propose a problem to study their behavior.

In Section 3, we prove Theorem 1.3. While working on this proof, we found a rather interesting relationship of the problem on Bohr's radius for wedge domains with a well-known problem on Brannan's polynomial coefficients, see [13,14,6,8], and mapping properties of hypergeometric functions. At the end of Section 3, we suggest a problem to prove some of these properties.

2. Bohr's radius for symmetric and alternating series

We start with two minor generalizations of Theorem 1.1, one for n-symmetric series and another for odd series. We recall that a function $f(z)$ analytic in $\mathbb{D}$ is called n-symmetric, where $n \geq 1$ is an integer, if $f(e^{2\pi i/n} z) = f(z)$ for all $z \in \mathbb{D}$. As is well known, $f(z)$ is n-symmetric if and only if its Taylor expansion has the following n-symmetric form:

$$f(z) = \sum_{k=0}^{\infty} a_{nk} z^{nk}.$$

Lemma 2.1. *If* $\left|\sum_{k=0}^{\infty} a_{nk} z^{nk}\right| \leq 1$ *in* $\mathbb{D}$, *then* $\sum_{k=0}^{\infty} |a_{nk}||z|^{nk} \leq 1$ *in the disk* $\mathbb{D}_{1/\sqrt[n]{3}}$. *The radius* $r = 1/\sqrt[n]{3}$ *is the best possible.*

Proof. Put $\zeta = z^n$ and consider a function $g(\zeta) = \sum_{k=0}^{\infty} a_{nk} \zeta^k$. This function is analytic in $\mathbb{D}$ and satisfies the inequality $|g(\zeta)| = |f(z)| \leq 1$ for all $|\zeta| < 1$. By Theorem 1.1, $\sum_{k=0}^{\infty} |a_{nk}||\zeta|^k \leq 1$ for all $|\zeta| \leq 1/3$ and therefore $\sum_{k=0}^{\infty} |a_{nk}||z|^{nk} \leq 1$ for all $|z| \leq 1/\sqrt[n]{3}$.

To show that the radius $1/\sqrt[n]{3}$ is the best possible, we modify the classical example used to prove sharpness of the radius $1/3$ in Theorem 1.1 (see [12,15]). Precisely, we consider the function $\varphi_{a,n}(z) = \varphi(z^n, a)$, where $\varphi(z, a) = (z - a)/(1 - az)$ with $0 < a < 1$. Then,

$$\varphi_{a,n}(z) = \frac{z^n - a}{1 - az^n} = -a + (1 - a^2)z^n \sum_{k=0}^{\infty} a^k z^{nk}. \tag{2.1}$$

Therefore,

$$M_{\varphi_{a,n}}(z) = a + (1 - a^2)r^n \sum_{k=0}^{\infty} a^k r^{nk} = \frac{a + r^n - 2a^2 r^n}{1 - ar^n}, \quad \text{where } r = |z|.$$

We claim that for every r such that $1/\sqrt[n]{3} < r < 1$ there is a such that $0 < a < 1$ and

$$\frac{a + r^n - 2a^2 r^n}{1 - ar^n} > 1. \tag{2.2}$$

Indeed, inequality (2.2) is equivalent to the inequality $P(a, r) < 0$, where $P(a, r) = 2a^2 r^n - a(1 + r^n) + 1 - r^n$. We have, $P(1, r) = 0$ and $\frac{\partial P}{\partial a}(1, r) = 3r^n - 1$. If $1/\sqrt[n]{3} < r < 1$, then $\frac{\partial P}{\partial a}(1, r) > 0$. The latter implies that $P(a, r) < 0$ and therefore $M_{\varphi_{a,n}}(z) > 1$ if $1/\sqrt[n]{3} < r < 1$ and if $a < 1$ is sufficiently close to 1. This shows that the radius $r = 1/\sqrt[n]{3}$ in Lemma 2.1 is the best possible. $\quad\square$

Proof of Theorem 1.2. Given a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $|f(z)| \leq 1$ for $|z| \leq 1$, consider its even and odd parts:

$$f_e(z) = \frac{1}{2}(f(z) + f(-z)) = \sum_{k=0}^{\infty} a_{2k} z^{2k},$$

$$f_o(z) = \frac{1}{2}(f(z) - f(-z)) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}.$$

Since $f(\mathbb{D}) \subset \mathbb{D}$ and since the unit disk $\mathbb{D}$ is a convex domain, which, in addition, is symmetric with respect to the origin, it follows that

$$f_e(z) \in \mathbb{D} \quad \text{and} \quad f_o(z) \in \mathbb{D} \quad \text{for all } z \in \mathbb{D}. \tag{2.3}$$

This implies, in particular, that the functions $f_e(z)$ and $f_o(z)$ each satisfies the hypothesis of Theorem 1.1.

Furthermore, for all $z \in \mathbb{D}$ the following relations hold true:

$$A_f(z) = \sum_{k=0}^{\infty} |a_{2k}||z|^{2k} - \sum_{k=0}^{\infty} |a_{2k+1}||z|^{2k+1} \tag{2.4}$$

$$\leq \sum_{k=0}^{\infty} |a_{2k}||z|^{2k} = A_{f_e}(z) = M_{f_e}(z) = M_{g_e}(\zeta),$$

where $\zeta = z^2$ and $g_e(\zeta) = \sum_{k=0}^{\infty} a_{2k} \zeta^k$.

The first relation in (2.3) implies that $|g_e(\zeta)| \leq 1$ for all $\zeta \in \mathbb{D}$. Thus, $g_e(\zeta)$ satisfies the hypothesis of Theorem 1.1 and therefore

$$M_{g_e}(\zeta) = \sum_{k=0}^{\infty} |a_{2k}||\zeta|^k \leq 1 \quad \text{for all } |\zeta| \leq \frac{1}{3}.$$

Combining this with equation (2.4) we conclude that $A_f(z) \leq 1$ for all $|z| \leq 1/\sqrt{3}$.

Actually, the proof presented above shows that if $f(z)$ is even and $|f(z)| \leq 1$ for all $|z| < 1$, then $M_f(z) = A_f(z) \leq 1$ for all $|z| \leq 1/\sqrt{3}$.

To find a lower bound for $A_f(z)$ we use a similar argument. We consider the following chain of relations:

$$A_f(z) = \sum_{k=0}^{\infty} |a_{2k}||z|^{2k} - \sum_{k=0}^{\infty} |a_{2k+1}||z|^{2k+1} \geq -\sum_{k=0}^{\infty} |a_{2k+1}||z|^{2k+1} \tag{2.5}$$

$$= A_{f_o}(z) = -M_{f_o}(z) = -|z|M_{g0}(z) > -M_{g_o}(z),$$

where $g_o(z) = f_o(z)/z$. Since $f_o(z)$ is odd it follows that the function $g_o(z)$ is even. Since $|f_o(z)| \leq 1$ for all $|z| < 1$ it follows that $|g_o(z)| \leq 1$ for all $|z| < 1$. Therefore, by our remark about even functions, $M_{g_o}(z) \leq 1$ for all $|z| \leq 1/\sqrt{3}$. Combining this with (2.5) we conclude that $A_f(z) \geq -M_{g_o}(z) \geq -1$ for all $|z| \leq 1/\sqrt{3}$. This, completes the proof of inequality (1.4).

We note that $A_f(z) = M_f(z)$ for even functions. Therefore, our example (2.1) with $n = 2$ shows that the radius $r = 1/\sqrt{3}$ is the best possible under the assumptions of Theorem 1.2. □

Example 2.1. One more example, which shows that the radius $r = 1/\sqrt{3}$ in Theorem 1.2 is the best possible, can be constructed by using the even part $\varphi_e(z)$ of the Möbius mapping

$$\varphi(z, a) = \frac{z + a}{1 + az}, \qquad 0 < a < 1, \tag{2.6}$$

which is given by

$$\varphi_e(z) = \frac{a(1 - z^2)}{1 - a^2 z^2} = a \left(1 - (1 - a^2)z^2 \sum_{k=0}^{\infty} a^{2k} z^{2k} \right).$$

Hence,

$$A_{\varphi_e} = a \left(1 + (1 - a^2)r^2 \sum_{k=0}^{\infty} a^{2k} r^{2k} \right) = a \frac{1 + (1 - 2a^2)r^2}{1 - a^2 r^2}, \quad \text{where } r = |z|.$$

We claim that for every r such that $1/\sqrt{3} < r < 1$ there is a such that $0 < a < 1$ and

$$a \frac{1 + (1 - 2a^2)r^2}{1 - a^2 r^2} > 1. \tag{2.7}$$

Indeed, inequality (2.7) is equivalent to the inequality

$$2a^3 r^2 - a^2 r^2 - a(1 + r^2) + 1 < 0. \tag{2.8}$$

Let $P_1(a, r)$ denote the left-hand side of (2.8). After elementary calculations, we find that $P_1(1, r) = 0$ and

$$\frac{\partial P_1}{\partial a}(1, r) = 3r^2 - 1.$$

The latter equation implies that $\frac{\partial P_1}{\partial a}(1, r) > 0$ if $1/\sqrt{3} < r < 1$. This inequality, combined with equation $P_1(1, r) = 0$, shows that if $1/\sqrt{3} < r < 1$ then (2.7) holds true for all $a < 1$ sufficiently close to 1. In particular, this shows once more that the radius $r = 1/\sqrt{3}$ in Theorem 1.2 is the best possible.

In the case $n = 2$, Lemma 2.1 gives Bohr's radius $r = 1/\sqrt{3}$ for the class of even functions. A modification of the proof of Lemma 2.1 gives also the following result for Bohr's radius for the class of odd functions.

Lemma 2.2. *Let $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$ be an odd analytic function in $\mathbb{D}$ such that $\left| \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} \right| \leq 1$ in $\mathbb{D}$.*

Then $\sum_{k=1}^{\infty} |a_{2k-1}| |z|^{2k-1} \leq 1$ in the disk $\mathbb{D}_{r_*}$, where r_* is a solution of the equation

$$5r^4 + 4r^3 - 2r^2 - 4r + 1 = 0,$$

which is unique in the interval $1/\sqrt{3} < r < 1$.

The value of r_* can be calculated in terms of radicals as

$$r_* = -\frac{1}{5} + \frac{1}{10}\sqrt{\frac{B+32}{3}} + \frac{1}{10}\sqrt{\frac{64}{3} - \frac{B}{3} + 144\sqrt{\frac{3}{B+32}}} = 0.7313\ldots, \tag{2.9}$$

where

$$B = 10 \cdot 2^{\frac{2}{3}} \left((47 - 3\sqrt{93})^{\frac{1}{3}} + (47 + 3\sqrt{93})^{\frac{1}{3}} \right).$$

Proof. First, we represent $f(z)$ as $f(z) = zg(z)$, where $g(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k}$. Then $g(z)$ is an even function such that $|g(z)| \leq 1$ in $\mathbb{D}$. It follows from Lemma 2.1 that

$$\sum_{k=0}^{\infty} |a_{2k+1}| |z|^{2k+1} \leq 1 \quad \text{for } |z| \leq 1/\sqrt{3}. \tag{2.10}$$

Our goal now is to show that inequality (2.10) holds on a larger interval. The following estimates were used by F. Wiener in his proof of Theorem 1.1 (see [11]). If $h(z) = \sum_{k=0}^{\infty} c_k z^k$ is analytic in $\mathbb{D}$ such that $|h(z)| \leq 1$ for all $z \in \mathbb{D}$, then

$$|c_k| \leq 1 - |c_0|^2, \quad k = 1, 2, \ldots \tag{2.11}$$

Applying (2.11) to the Taylor coefficients of $g(z)$, we obtain the following:

$$\sum_{k=0}^{\infty} |a_{2k+1}| |z|^{2k+1} \leq r \left(a + (1 - a^2) \sum_{k=1}^{\infty} r^{2k} \right) = r \left(a + (1 - a^2) \frac{r^2}{1 - r^2} \right),$$

where $r = |z|$ and $a = |a_1|$.

Now we want to find the largest interval $1/\sqrt{3} \leq r \leq r_0$ such that

$$\frac{r(a(1 - r^2) + (1 - a^2)r^2)}{1 - r^2} \leq 1$$

for all $0 < a < 1$.

The latter inequality is equivalent to the inequality

$$P_2(a, r) = a^2 r^3 - ar(1 - r^2) + 1 - r^2 - r^3 \geq 0.$$

Since $P_2(a, r)$ is quadratic in a its minimum value $P_2^{min}(r)$ is achieved at $a = (1 - r^2)/2r^2$ and is equal to $-P_2(r)/4r$, where

$$P_2(r) = 5r^4 + 4r^3 - 2r^2 - 4r + 1.$$

Since

$$P_2''(r) = 4(15r^2 + 6r - 1) > 0$$

for $1/\sqrt{3} \le r < 1$ the function $P_2(r)$ is convex on the interval $1/\sqrt{3} < r < 1$. Since $P_2(1/\sqrt{3}) = \frac{8}{9}(1 - \sqrt{3}) < 0$, $P_2(1) = 4 > 0$ it follows that the equation $P_2(r) = 0$, or equivalently equation $P_2^{min}(r) = 0$, has exactly one solution r_* on $1/\sqrt{3} < r < 1$. One may use *Mathematica* or *Maple* to verify that this solution is explicitly given by the equation (2.9).

Since

$$P_2(a, r) \ge P_2^{min}(r) = -P_2(r)/4r > 0$$

for all r and a such that $1/\sqrt{3} \le r < r_*$, $0 < a < 1$, the required inequality $\sum_{k=1}^{\infty} |a_{2k-1}||z|^{2k-1} \le 1$ holds for $|z| < r_*$.   □

To find an upper bound for Bohr's radius for the class of odd functions discussed in Lemma 2.2, we modify our basic example used to show sharpness in Lemma 2.1.

Example 2.2. Consider the odd function

$$\psi_a(z) = z \frac{z^2 - a}{1 - az^2} = z\left(-a + (1 - a^2)z^2 \sum_{k=0}^{\infty} a^k z^{2k}\right), \qquad 0 < a < 1.$$

We have

$$M_{\psi_a}(r) = r\left(a + (1 - a^2)r^2 \sum_{k=0}^{\infty} a^k r^{2k}\right) = r\frac{a + r^2 - 2a^2 r^2}{1 - ar^2}.$$

The inequality $M_{\psi_a}(r) > 1$ is equivalent to the inequality

$$2a^2 r^3 - ar(1 + r) + 1 - r^3 < 0. \tag{2.12}$$

We want to find all r, $1/\sqrt{3} < r < 1$, such that for each of these r there is a, $0 < a < 1$, such that (2.12) holds.

Let $P_3(a, r)$ denote the left-hand side of the inequality (2.12). The minimum value of $P_3(a, r)$ considered as a function of a occurs at

$$a_3^{min} = \frac{1 + r}{4r^2}.$$

Notice that $0 < a_3^{min} < 1$ for $0.6403\ldots = \frac{1+\sqrt{17}}{8} < r < 1$. Thus, $\min_{0 < a < 1} P_3(a, r) = -\frac{1}{8r} P_3(r)$, where

$$P_3(r) = 8r^4 + r^2 - 6r + 1.$$

One can easily check that $P_3(0) > 0$, $P_3(1) > 0$, $P_3''(r) > 0$ for $0 < r < 1$, and that $P_3(r)$ has two zeros on the interval $0 < r < 1$. Let r^* denote the larger zero; its expression in terms of radicals can be found with *Mathematica* which gives:

$$r^* = \frac{1}{4}\sqrt{\frac{BC - 2}{6}} + \frac{1}{2}\sqrt{3\sqrt{\frac{6}{C - 2}} - \frac{C}{24} - \frac{1}{6}},$$

where

$$C = (3601 - 192\sqrt{327})^{\frac{1}{3}} + (3601 + 192\sqrt{327})^{\frac{1}{3}}.$$

Calculating r^* numerically, we find $r^* = 0.7899\ldots > r_*$, where $r_* = 0.7313\ldots$ is given in Lemma 2.2. Thus, we have shown that for every r such that $r^* \leq r < 1$ there is a, $0 < a < 1$, such that $M_{\psi_a}(r) > 1$.

Combining the upper bound r^* given in Example 2.2 with the lower bound r_* provided by Lemma 2.2, we conclude that Bohr's radius for the class of odd functions satisfies the inequalities $r_* \leq r \leq r^*$. There is a gap between these lower and upper bounds and therefore the following problem remains open.

Problem 2.1. Find Bohr's radius for the class of odd functions $f(z)$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$.

Remark 2.1. Among the interesting results in [15], the authors found that Bohr's radius can be improved from $1/3$ to $1/2$ for analytic functions satisfying the additional condition $f(0) = 0$. Of course, $f(0) = 0$ if $f(z)$ is an odd function. Thus, we note here that our lower bound $r_* = 0.7313...$ is significantly better than $1/2$.

Remark 2.2. We have shown in Example 2.1 that the even part of function (2.6) produces the sharp upper bound for Bohr's radius for the class of even functions. We tried the same idea hoping to improve our upper bound for Bohr's radius for odd functions given in Example 2.2. Thus, we tested the odd part of the function (2.6), which is

$$\varphi_o(z, a) = \frac{(1 - a^2)z}{1 - a^2 z^2} = (1 - a^2)z \sum_{k=0}^{\infty} a^{2k} z^{2k},$$

and we found that $M_{\varphi_o}(z) = \frac{(1-a^2)r}{1-a^2 r^2} < 1$ for all $0 < r < 1$ and all $0 < a < 1$. Thus, this example gives only E trivial upper bound in the case under consideration.

The majorant and alternating series defined by formulas (1.2) and (1.3) for an analytic function $f(z)$ in (1.1) are special cases of a more general type of series associated with $f(z)$ which can be defined for all positive integers n by

$$S_f^n(z) = \sum_{k=0}^{\infty} e^{\frac{2\pi i k}{n}} |a_k| |z|^k. \tag{2.13}$$

Then we obviously have

$$M_f(z) = S_f^1(z) \quad \text{and} \quad A_f(z) = S_f^2(z).$$

Arguments of coefficients of series (2.13) are equally spaced over the interval $[0, 2\pi)$. Thus, $S_f^n(z)$ can be considered as a kind of *argument symmetric series* associated with $f(z)$. We are not aware of any practical use of such argument symmetric series and thus our next problem is posed just out of curiosity.

Problem 2.2. Given a positive integer $n \geq 2$, find the largest radius r_n such that $|S_f^n(z)| \leq 1$ for all $|z| \leq r_n$ whenever $|f(z)| \leq 1$ for all $z \in \mathbb{D}$.

In particular, it would be interesting to know whether the sequence r_n with $n = 2, 3, \ldots$ is monotonic. Since, $|S_f^n(z)| \leq M_f(z)$ it follows that $r_n \geq r_1 = 1/3$ for all $n \geq 2$. Also, an upper bound for $S_f^n(z)$ provided by the function $\varphi(z, a) = (z - a)/(1 - az)$ shows that $r_n \to r_1 = 1/3$ as $n \to \infty$.

One more series with evenly distributed arguments of coefficients associated with the function $f(z)$ can be defined by

$$T_f^n(z) = \sum_{k=0}^{\infty} e^{\frac{2\pi i k}{n}} |a_k| z^k. \tag{2.14}$$

Although series (2.14) might present an interest in connection with some problems on the class of bounded analytic functions, it does not provide new bounds for the sup norms of $f(z)$. Indeed, taking $z = re^{-\frac{2\pi i}{n}}$ one can see that $T_f^n(z) = M_f(z)$ in this case.

3. Bohr's radius for wedge mappings

In the beginning of this section, we recall some results needed for the proof of Theorem 1.3. For $a \in W_\alpha$, let $F_a(z)$ denote the Riemann mapping function from $\mathbb{D}$ onto W_α such that $F_a(0) = a$, $F_a'(0) > 0$. This function can be expressed in the form

$$F_a(z) = a F_{\alpha,\gamma}(e^{i\gamma(1-\alpha)} z) \tag{3.1}$$

with $\gamma = (\arg a)/\alpha$, where the function $F_{\alpha,\gamma}(z)$ is given by

$$F_{\alpha,\gamma}(z) = \left(\frac{1 + e^{-2i\gamma} z}{1 - z} \right)^\alpha. \tag{3.2}$$

Let $F_a(z) = a + \sum_{n=1}^{\infty} A_n z^n$ be the Taylor expansion of $F_a(z)$ at $z = 0$. First, we will discuss properties of the coefficients A_n needed for our proof. It follows from (3.1) that $|A_n| = |a||A_n(\alpha, \gamma)|$, where $A_n(\alpha, \gamma)$ are the Taylor coefficients in the expansion

$$F_{\alpha,\gamma}(z) = 1 + \sum_{n=1}^{\infty} A_n(\alpha, \gamma) z^n. \tag{3.3}$$

We recall here that the function $F_{\alpha,\gamma}(z)$ and its coefficients $A_n(\alpha, \gamma)$ play an important role in the study of functions with bounded boundary rotation, see [6,14,13,8] where some explicit forms of these coefficients were used.

The following two lemmas are key ingredients of our proof of Theorem 1.3. The first lemma follows from a well-known result of Y. Abu-Muhanna and D. Hallenberg [2] on subordination of functions mapping the unit disk into domains having convex complement, which contains the origin. We remind the reader that a function $g(z)$ analytic in the unit disk $\mathbb{D}$ is subordinate in $\mathbb{D}$ to a function $f(z)$ if there is a function $\varphi(z)$ analytic in $\mathbb{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$ so that $g(z) = f(\varphi(z))$.

Lemma 3.1. *Let $1 \leq \alpha \leq 2$, $a \in W_\alpha$, $\Im a \geq 0$, and let $f(z) = a + \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ such that $f(\mathbb{D}) \subset W_\alpha$. Then*

$$|a_n| \leq |A_n| = |a||A_n(\alpha, \gamma)|, \quad k = 1, 2, \ldots \tag{3.4}$$

Proof. Inequality (3.4) with $a > 0$ was used in a recent paper [4]. For completeness, we present its short proof valid for any $a \in W_\alpha$. The function $F_a(z)$ maps $\mathbb{D}$ conformally and one-to-one onto the wedge domain W_α. Thus, the complement $\mathbb{C} \setminus W_\alpha$ is convex and $0 \in \mathbb{C} \setminus W_\alpha$. Furthermore, the function $f(z)$ is subordinate to $F_a(z)$. Then by Theorem 1 in [2], $f(z)$ admits the integral representation

$$f(z) = \int_{|x|=1} F_a(xz) \, d\mu(x), \tag{3.5}$$

where μ is a probability measure on $|x| = 1$. This equation can be rewritten in terms of Taylor series as

follows,

$$a + a_1 z + \ldots + a_n z^n + \ldots = \int\limits_{|x|=1} (a + A_1 zx + \ldots + A_n z^n x^n + \ldots) \, d\mu(x).$$

Multiplying the latter equation by $\bar{z}^n = r^n e^{-in\theta}$ with $0 < r < 1$, and then integrating term by term over the interval $0 \le \theta \le 2\pi$, we obtain the following equation involving the Taylor coefficient a_n of $f(z)$ and Taylor coefficient A_n of $F_a(z)$:

$$a_n = \int\limits_{|x|=1} A_n x^n \, d\mu(x).$$

Applying the triangle inequality for the integrals to this equation, we obtain

$$|a_n| \le \int\limits_{|x|=1} |A_n||x^n| \, d\mu(x) = |A_n|, \tag{3.6}$$

which is the required inequality (3.4). $\square$

Remark 3.1. We note that equality in (3.6) occurs if and only if the argument of $A_n x^n$ is constant for all x in the support of the measure μ. This implies that equality occurs in the equation (3.6), or equivalently in the equation (3.4), if and only if $f(z)$ can be represented by formula (3.5) with a discrete measure μ with support on the set $\{x = e^{i(\theta_0 + \frac{2\pi k}{n})} : k = 0, \ldots, n-1\}$ with some θ_0, $0 \le \theta_0 < 2\pi$. In particular, $|a_1| = |A_1|$ if and only if μ in (3.5) is a Dirac measure supported at a point $x = e^{i\theta_0}$, $0 \le \theta_0 < 2\pi$.

The second key lemma contains inequalities conjectured by D. Brannan and proved by D. Aharonov and S. Friedland [6]. After that a shorter proof was given by Brannan in [13].

Lemma 3.2 ([6,13]). *Let $1 \le \alpha \le 2$ and let $A_n(\alpha, \gamma)$ be coefficients defined by (3.3). Then*

$$|A_n(\alpha, \gamma)| < A_n(\alpha, 0) \tag{3.7}$$

for all $0 < \gamma < \pi/2$ and all $n = 1, 2, \ldots$.

Proof of Theorem 1.3. Let $f(z)$ satisfy the conditions of the theorem. First, we note that without loss of generality we may assume that $dist(a_0, \partial W_\alpha) = 1$. Indeed, $f(z)$ satisfies (1.6) for some $z \in \mathbb{D}$ if and only if the modified function $\widetilde{f}(z) = f(z)/dist(a_0, \partial W_\alpha)$ satisfies the same type of inequality for the same z. Furthermore, dilations preserve sectors with the vertex at the origin. Thus, $\widetilde{f}(\mathbb{D}) \subset W_\alpha$ and therefore $\widetilde{f}(z)$ also satisfies the assumptions of the theorem. Therefore, in proving the theorem, we may work with $\widetilde{f}(z)$ instead of $f(z)$.

Next we consider the set E of all points $a \in W_\alpha$ such that $dist(a, \partial W_\alpha) = 1$. This set is shown in Fig. 1, which also illustrates some other notations of this proof. The set E consists of the points of the arc $C_\alpha = \{e^{i\theta} : |\theta| < \pi(\alpha - 1)/2\}$ and the points of the rays $R_\alpha^+ = \{w = e^{i\pi(\alpha-1)/2} + t e^{i\pi\alpha/2} : t \ge 0\}$ and $R_\alpha^- = \{w : \overline{w} \in R_\alpha^+\}$.

Let $B_n(t)$ denote the n-th coefficient of the Taylor expansion of $F_{a(t)}(z)$ with $a(t) = e^{i\pi(\alpha-1)/2} + t e^{i\pi\alpha/2} \in R_\alpha^+$. In the case $a(t) \in R_\alpha^-$, the proof follows same lines and therefore is omitted. We claim that the modulus $|B_n(t)|$ is a non-increasing function of t, $t \ge 0$. To prove this, let $0 \le t_1 < t_2$ and consider the shifted function $\widetilde{F}(z) = F_{a(t_2)}(z) - (t_2 - t_1)e^{i\pi\alpha/2}$. Then, $\widetilde{F}(0) = a(t_1)$ and $\widetilde{F}(\mathbb{D}) \subset W_\alpha$. Thus, $\widetilde{F}(z)$ is subordinate to $F_{a(t_1)}(z)$ and $\widetilde{F}(\mathbb{D})$ is a proper subset of W_α. Therefore, it follows from Lemma 3.1 that
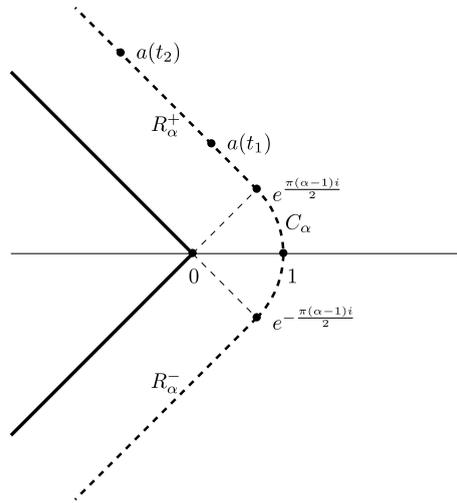
Fig. 1. Set $E = \{a \in W_\alpha : \ dist(a, \partial W_\alpha) = 1\}$.

$$|B_n(t_2)| \leq |B_n(t_1)| \quad \text{for all } n \geq 1.$$

Thus, the required monotonicity is proved and therefore

$$|B_n(t)| \leq |B_n(0)| = |A_n(\alpha, \pi(\alpha - 1)/(2\alpha))| \tag{3.8}$$

for all $n \geq 1$ and all $t > 0$.

Now let $a = a(\tau) = e^{i\tau}$, $|\tau| \leq \pi(\alpha-1)/2$ and let $\widetilde{B}(\tau)$ denote the n-th coefficient of the Taylor expansion of $F_{a(\tau)}(z)$ considered as a function of τ. Without loss of generality we may assume that $0 \leq \tau \leq \pi(\alpha-1)/2$. In case $-\pi(\alpha - 1)/2 \leq \tau \leq 0$ the proof follows same lines. It follows from (3.7) that

$$|\widetilde{B}(\tau)| < A_n(\alpha, 0) \tag{3.9}$$

for all τ, $0 < \tau \leq \pi(\alpha - 1)/2$. This together with (3.8) shows also that

$$|B_n(t)| < A_n(\alpha, 0) \tag{3.10}$$

for all $t \geq 0$.

Suppose now that $f(z) = a + a_1 z + \ldots + a_n z^n + \ldots$ satisfies the assumptions of the theorem with some $a \in E$. Then, by Lemma 3.1, $|a_n| \leq B_n(t)$ for all $n \geq 1$ if $a = a(t) \in R_\alpha^+$ or $|a_n| \leq |\widetilde{B}(\tau)|$ for all $n \geq 1$ if $a = a(\tau) = e^{i\tau} \in C_\alpha$. Combining the latter inequalities for coefficients a_n with inequalities (3.9) and (3.10), we find that

$$|a_n| \leq A_n(\alpha, 0) \quad \text{for all } n \geq 1. \tag{3.11}$$

Using inequalities (3.11), obtained for individual coefficients a_n, we conclude that for $f(z)$ satisfying the assumptions of the theorem and such that $dist(a_0, \partial W_\alpha) = 1$ the following inequality holds:

$$\sum_{k=1}^{\infty} |a_k||z|^k \leq \sum_{k=1}^{\infty} A_k(\alpha, 0)|z|^k = F_{\alpha,0}(r) - 1 = \left(\frac{1+r}{1-r}\right)^\alpha - 1, \tag{3.12}$$

where $r = |z|$.

Now, an elementary check shows that

$$\left(\frac{1+r}{1-r}\right)^{\alpha} - 1 \leq 1$$

for all r such that $0 \leq r \leq r_{\alpha}$ where r_{α} is defined in Theorem 1.3 and that this inequality is not valid for $r_{\alpha} < r < 1$. The latter being combined with inequality (3.12) proves (1.6) and shows that the radius r_{α} is the best possible. This completes the proof of Theorem 1.3. $\quad\square$

Our proof of Theorem 1.3 depends heavily on the Aharonov–Friedland inequalities (3.7) for the coefficients $A_n(\alpha, \gamma)$ of the function $F_{\alpha,\gamma}(z)$ defined by (3.2). These coefficients are a special case of the so-called *Brannan's coefficients*, which found important applications in the study of functions with bounded boundary rotation and related questions (see [13,16,8]).

Brannan's coefficients can be explicitly expressed in terms of hypergeometric functions as is shown below. To simplify notation, we will introduce the new parameters c and $s = c - 1$, which are more convenient for our purposes. First we let

$$c = 2e^{-i\gamma}\cos\gamma = 1 + e^{-2i\gamma}.$$

Then, $F_{\alpha,\gamma}(z)$ can be written in the form:

$$F_{\alpha,\gamma}(z) = \left(1 + c\frac{z}{1-z}\right)^{\alpha}.$$

For $\alpha = 1$, the function $F_{\alpha,\gamma}(z)$ is fractional linear. Excluding this trivial case, we will assume below that $1 < \alpha \leq 2$.

Using the binomial formula, we obtain

$$\left(1 + c\frac{z}{1-z}\right)^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}c^k\left(\frac{z}{1-z}\right)^k = \sum_{n=0}^{\infty}C_n z^n,$$

where $C_0 = 1$ and

$$C_n = \sum_{k=1}^{n}\binom{\alpha}{k}\binom{n-1}{n-k}c^k = (\alpha c) \cdot {}_2F_1(1-\alpha, 1-n; 2; c), \quad n \geq 1,$$

where notation ${}_2F_1$ stands for the Gauss hypergeometric function.

It will be convenient to replace c with $1 + s$ and treat s as a complex variable in the closed unit disk. Thus, we assume that $s = re^{i\theta} \in \overline{\mathbb{D}}$. With this notation, coefficients C_n become functions of α and s given by

$$
\begin{aligned}
C_n(\alpha, s) &= \sum_{k=1}^{n}\left[\binom{\alpha}{k}\binom{n-1}{n-k}\sum_{j=0}^{k}\binom{k}{j}s^j\right]\\
&= \sum_{m=0}^{n}\frac{\alpha\Gamma(n+\alpha-m)}{\Gamma(m+1)\Gamma(1+\alpha-m)\Gamma(n+1-m)}s^m\\
&= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\,{}_2F_1(-\alpha, -n; 1-\alpha-n; -s).
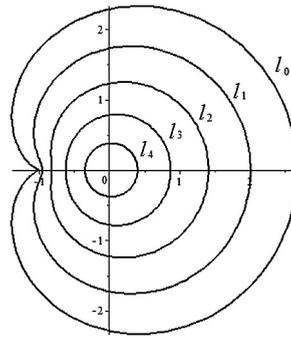\end{aligned}
$$

Fig. 2. Images of circles for $_2F_1(-\alpha, -n; 1 - \alpha - n; re^{i\theta}) - 1$.

For $s = e^{-2i\gamma}$ with $0 \leq \gamma < \pi/2$, the Aharonov–Friedland inequality (3.7) is equivalent to the inequality

$$|_2F_1(-\alpha, -n; 1 - \alpha - n; -s)| < |_2F_1(-\alpha, -n; 1 - \alpha - n; -1)| \quad \text{fro } \alpha \geq 1, \, n > 1.$$

We finish this section by proposing a problem, supported by numerical evidence, to prove related monotonicity properties of the function $_2F_1(-\alpha, -n; 1 - \alpha - n; z)$.

Problem 3.1. Prove the following:

(a) The modulus of the hypergeometric polynomial $_2F_1(-\alpha, -n; 1 - \alpha - n; z)$ with $z = e^{i\theta}$ is an increasing function of θ in $0 \leq \theta \leq \pi$.
(b) The function $F(z) = {}_2F_1(-\alpha, -n; 1 - \alpha - n; z) - 1$ with $z = re^{i\theta}$ maps $\mathbb{D}$ conformally and one-to-one onto a domain circularly symmetric with respect to the positive real axis. (We recall that a domain D is called circularly symmetric with respect to the positive real axis if for every $r > 0$ the intersection $D \cap \{z : |z| = r\}$ is either a circle, or empty, or a circular arc having its middle point at $z = r$.)
(c) If $z \in \mathbb{D}$ and $\Im z > 0$, then

$$\Im \frac{z \cdot {}_2F_1(1 - \alpha, 1 - n; 2 - \alpha - n; z)}{_2F_1(-\alpha, -n; 1 - \alpha - n; z) - 1} > 0. \tag{3.13}$$

The left-hand side of the inequality (3.13) was obtained by differentiation of the function $\log |_2F_1(-\alpha, -n; 1 - \alpha - n; re^{i\theta})|$ with respect to θ. Thus, the inequality (3.13), if true, will imply parts **(a)** and **(b)** as well. Numerical evidence supports monotonicity properties stated in Problem 3.1. In particular, Fig. 2 displays the nice monotonic behavior of images l_k of the circles $\{z : |z| = 1 - 0.2k\}$ for $k = 0, 1, 2, 3, 4$ after a mapping by the function $_2F_1(-\alpha, -n; 1 - \alpha - n; z) - 1$ with $\alpha = 1.9$ and $n = 50$.

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